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Topological Field Theories Associated with Three-Dimensional Seiberg–Witten Monopoles

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Three-dimensional topological field theories associated with the three-dimensional version of Abelian and non-Abelian Seiberg–Witten monopoles are presented. These three-dimensional monopole equations are obtained by a dimensional reduction of the four-dimensional ones. The starting actions to be considered are Gaussian types with random auxiliary fields. As the local gauge symmetries with topological shifts are found to be first-stage-reducible, the Batalin–Vilkovisky algorithm is suitable for quantization. Then the BRST transformation rules are automatically obtained. Nontrivial observables associated with Chern classes are obtained from the geometric sector and are found to correspond to those of the topological field theory of Bogomol'nyi monopoles.

1. INTRODUCTION

Topological field theories (Schwarz, 1978; Witten, 1988; Birmingham *et al.*, 1991; Thompson, 1993) are often used to study the topological nature of manifolds. In particular, three- and four-dimensional topological field theories are well developed. The best known three-dimensional topological field theory is the Chern–Simons theory, whose partition function gives the Ray–Singer torsion of three–manifolds (Schwarz, 1978), and the other topological invariants can be obtained as gauge-invariant observables, i.e., Wilson loops. The correlation functions can be identified with knot or link invariants, e.g., the Jones polynomial or its generalizations. On the other hand, in four dimensions, the twisted $N = 2$ supersymmetric Yang–Mills theory developed by Witten (1988) also has the nature of topological field

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theory. This Yang–Mills theory can be interpreted as a Donaldson theory (Donaldson, 1983) and the correlation functions are identified with Donaldson polynomials which classify smooth structures of topological four-manifolds. However, a new topological field theory on four-manifolds was discovered in recent studies of the electric–magnetic duality of supersymmetric gauge theory. The story of this development is as follows.

Seiberg and Witten (1994a, b) studied the electric–magnetic duality of $N = 2$ supersymmetric $SU(2)$ Yang–Mills gauge theory (for reviews, see Di Vecchia, 1996; Labastida, 1995; Intriligator and Seiberg, 1996; Thompson, 1995; Witten, 1995) by using a version of Montonen–Olive duality, and obtained exact solutions. According to this result, the exact low-energy effective action can be determined by a certain elliptic curve with a parameter $u = \langle \mathrm{tr}\phi^2 \rangle$, where ϕ is a complex scalar field in the adjoint representation of the gauge group, describing the quantum moduli space. For large u, the theory is weakly coupled and semiclassical, but at $u = \pm\Lambda^2$ corresponding to the strong-coupling regime, where Λ is the dynamically generated mass scale, the elliptic curve becomes singular and the situation of the theory changes drastically. At these singular points, magnetically charged particles become massless. Witten showed that at $u = \pm\Lambda^2$ the topological quantum field theory was related to the moduli problem of counting the solutions of the (Abelian) "Seiberg–Witten monopole equations" (Witten, 1994a) and it gave a dual description for the $SU(2)$ Donaldson theory. The particularly interesting fact is that the partition function of this $U(1)$ gauge theory produces a new topological invariant (Kronheimer and Mrowka, 1994; Taubes, 1994; Witten, 1994a, b; Akbulut, 1995; Bradlow and Garcia-Prada, 1996; Donaldson, 1996).

The topological field theory of the Seiberg–Witten monopoles has been discussed by several authors. Labastida and Mariño (1995a), using the Mathai–Quillen formalism (Mathai and Quillen, 1986; Atiyah and Jeffrey, 1990; Blau, 1993), found that the resulting action was equivalent to that of the twisted $N = 2$ supersymmetric Maxwell coupled with a twisted $N = 2$ hypermultiplet. Furthermore, they generalized their results for non-Abelian cases (Labastida and Mariño, 1995b, c) and determined polynomial invariants for the $SU(2)$ case corresponding to a generalization of Witten (1994a, b) in the Abelian case. In these studies, the topological field theories were formulated as of Witten type. On the other hand, Hyun *et al.* (1995a, b) discussed a non-Abelian topological field theory in view of twisting of $N = 2$ supersymmetric Yang–Mills coupled with $N = 2$ matter and obtained similar polynomial invariants. There are other approaches to obtaining the topological action; in fact, Carey *et al.* (1997) derived the topological action as a BRST variation of a certain gauge fermion, Gianvittorio *et al.* (1996, 1997) discussed in view of a covariant gauge-fixing procedure.

In three dimensions, a topological field theory of Bogomol'nyi monopoles can be obtained from a dimensional reduction of Donaldson theory (Baulieu and Grossman, 1988; Birmingham *et al.*, 1989) and the partition function of this theory gives the Casson invariant (Atiyah and Jeffrey, 1990). However, the three- or two-dimensional topological field theory of Seiberg–Witten monopoles does no seem to have been fully discussed, although several authors point out its importance (Thompson, 1995; Olsen, 1996; Carey *et al.*, 1997). Carey *et al.* (1997) performed a dimensional reduction of the Abelian Seiberg–Witten theory from four to three dimensions and found the reduced topological action. They also found, in view of the Mathai–Quillen formalism, that the partition function of this three-dimensional theory can be interpreted as a Seiberg–Witten version of the Casson invariant of three-manifolds.

In this paper, we discuss the topological quantum field theories associated with the three-dimensional version of Abelian and non-Abelian Seiberg–Witten monopoles, by applying Batalin–Vilkovisky quantization. In particular, we construct the topological actions, topological observables, and BRST transformation rules. In Section 2, we briefly review the essence of topological quantum field theories of both Witten type and Schwarz type. The reader interested in the results of this paper may neglect this section. In Section 3, the dimensional reduction of the Abelian and non-Abelian Seiberg–Witten monopole equations is explicitly performed and the three-dimensional monopole (3D monopole) equations are obtained. We also obtain quadratic actions which reproduce these three-dimensional monopole equations as minimum. In Section 4, we construct topological field theories of these three-dimensional monopoles, taking the actions including random auxiliary fields as a starting point. As their local gauge symmetries are classified as first-stage-reducible with on-shell reducibility, the Batalin–Vilkovisky algorithm is suitable to quantize these theories. Then we can automatically obtain the BRST transformation rules by construction. It is shown that the observables in the geometric sector can be obtained in a standard fashion, but those in the matter sector are found to be trivial. Our results for the Abelian case are consistent with those of the dimensionally reduced version of the topological field theory of four-dimensional Seiberg–Witten monopoles (Carey *et al.*, 1997), while those for the non-Abelian case are new results. It is interesting to compare our results with those of the topological field theory of Bogomol'nyi monopoles. Section 5 is a summary, and we also mention some open problems.

Notations

We use the following notations, unless mentioned otherwise. Let X be a compact orientable spin four–manifold with no boundary and let $g_{\mu\nu}$ be its

Riemannian metric tensor with $g = \det g_{\mu\nu}$. We use x_μ as the local coordinates on X. The γ_μ are Dirac gamma matrices and $\sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]/2$ with $\{\gamma_\mu, \gamma_\nu\} = g_{\mu\nu}$ (see also the Appendix). M is a Weyl fermion and $\bar{M}$ is the complex conjugate of M. We suppress spinor indices. The Lie algebra **g** is defined by $[T^a, T^b] = if_{abc}T^c$, where T^a is a generator normalized as tr $T^aT^b = \delta^{ab}$. The symbol f_{abc} is the structure constant of **g** and is antisymmetric in its indices.

The Greek indices μ, ν, α, . . . run from 0 to 3. The Roman indices a, b, c, . . . are used for the Lie algebra indices running from 1 to dim **g**, whereas i, j, k, . . . are the indices for space coordinates. Space-time indices are raised and lowered with $g_{\mu\nu}$. The repeated indices are assumed to be summed. $\varepsilon_{\mu\nu\rho\sigma}$ is an antisymmetric tensor with $\varepsilon_{0123} = 1$. We often use the abbreviation of Roman indices as $\theta = \theta^aT^a$, etc., in order to suppress the summation over Lie algebra indices.

2. QUICK TOUR OF TOPOLOGICAL FIELD THEORY

This section is devoted to a brief review of topological field theory. The reader interested in the details should refer to Schwarz (1978), Witten (1988), Birmingham *et al.* (1991), Thompson (1993), and Labastida (1995).

Let ϕ be any field content. For a local symmetry of ϕ, we can construct a nilpotent BRST operator Q_B ($Q_B^2 = 0$). The variation of any functional $\mathbb{O}$ of ϕ is denoted by

$$\delta\mathbb{O} = \{Q_B, \mathbb{O}\} \tag{2.1}$$

where the bracket $\{*, *\}$ means a graded commutator, namely, if $\mathbb{O}$ is bosonic, the bracket means a commutator $[*, *]$, and otherwise it is an antibracket.

Then we can give the definition of topological field theory (Birmingham *et al.*, 1991).

Definition. A topological field theory consists of:

1. a collection of Grassmann graded fields ϕ on an n-dimensional Riemannian manifold X with a metric g;
2. a nilpotent Grassmann odd operator Q;
3. physical states to be Q-cohomology classes;
4. an energy-momentum tensor $T_{\alpha\beta}$ which is Q-exact for some functional $V_{\alpha\beta}$ such that

$$T_{\alpha\beta} = \{Q, V_{\alpha\beta}(\phi, g)\} \tag{2.2}$$

In this definition, Q is often identified with Q_B and is in general independent of the metric. There are several examples of topological field theories which do not satisfy this definition, but this definition is useful in many cases.

There are two broad types of topological field theories satisfying this definition, Witten type (Witten, 1988) or Schwarz type (Schwarz, 1978) (there are several nonstandard Schwarz-type theories, e.g., higher dimensional BF theories, but here we do not consider such cases).

For the Witten-type theory, the quantum action S_q which comprises the classical action, ghost, and gauge-fixing terms can be represented by $S_q = \{Q_B, V\}$ for some function V of metric and fields and BRST charge Q_B. Under the metric variation δ_g of the partition function Z, it is easy to see that

$$\begin{aligned} \delta_g Z &= \int \mathcal{D}\phi\, e^{-S_q}\left(-\frac{1}{2}\int_X d^n x \sqrt{g}\, \delta g^{\alpha\beta} T_{\alpha\beta}\right) \\ &= \int \mathcal{D}\phi\, e^{-S_q}\{Q, \chi\} \\ &\equiv \langle\{Q, \chi\}\rangle = 0 \end{aligned} \tag{2.3}$$

where

$$\chi = -\frac{1}{2}\int_X d^n x \sqrt{g}\, \delta g^{\alpha\beta} V_{\alpha\beta} \tag{2.4}$$

The last equality in (2.3) follows from the BRST invariance of the vacuum and means that Z is independent of the local structure of X, that is, Z is a "topological invariant" of X.

In general, for a Witten-type theory, Q_B can be constructed by the introduction of a topological shift with other local gauge symmetry (Baulieu and Singer, 1988; Brooks *et al.*, 1988). For example, in order to obtain the topological Yang–Mills theory on the four-manifold M^4, we introduce a shift in the gauge transformation for the gauge field A^a_μ such that $\delta A^a_\mu = D_\mu \theta^a + \varepsilon^a_\mu$, where D_μ is the covariant derivative, and θ^a and ε^a_μ are the (Lie-algebra-valued) usual gauge transformation parameter and topological shift parameter, respectively. In order to see the role of this shift, let us consider the first Pontrjagin class on M^4 given by

$$S = \frac{1}{8}\int_{M^4} \varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}\, d^4x \tag{2.5}$$

where $F^a_{\mu\nu}$ is the field strength of the gauge field. We can easily check the invariance of (2.5) under the action of δ. In this sense, (2.5) has a larger symmetry than the usual (Yang–Mills) gauge symmetry. Taking this into account, we can construct the topological Yang–Mills gauge theory (Baulieu and Singer, 1988; Brooks *et al.*, 1988; Gomis *et al.*, 1995). We can also consider similar "topological" shifts for matter fields, as will be shown in Section 4.

In addition, in general, Witten-type topological field theory can be obtained from the quantization of certain Langevin equations (Birmingham *et al.*, 1991). This approach has been used for the construction of several topological field theories, e.g., supersymmetric quantum mechanics, topological sigma models, or Donaldson theory (Labastida and Pernici, 1988; Birmingham *et al.*, 1991) [we will use this approach for the $N = 4$ theory (Vafa and Witten, 1994) elsewhere (Ohta, n.d.)].

On the other hand, Schwarz-type theory (Schwarz, 1978) begins with any metric-independent classical action S_c as a starting point, but S_c is assumed not to be a total derivative. Then the quantum action (up to gauge-fixing term) can be written as

$$S_q = S_c + \{Q, V(\phi, g)\} \tag{2.6}$$

for some function V. For this quantum action, we can easily check the topological nature of the partition function, but note that the energy-momentum tensor contributes only from the second term in (2.6). One of the differences between Witten-type and Schwarz-type theories can be seen in this point. Namely, the energy-momentum tensor of the classical action for Schwarz-type theory vanishes because it is derived as a result of metric variation.

Finally, we comment on the local symmetry of Schwarz-type theory. Let us consider the Chern–Simons theory as an example. The classical action

$$S_{\mathrm{CS}} = \int_{M^3} d^3x \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{2.7}$$

is a topological invariant which gives the second Chern class of the three-manifold M^3. As is easy to find, S_{CS} is not invariant under the topological gauge transformations, although it is (Yang–Mills) gauge-invariant. Therefore the quantization proceeds by the standard BRST method. This is a general feature of Schwarz-type theory.

3. DIMENSIONAL REDUCTION

In this section, the dimensional reduction of the Abelian and non-Abelian Seiberg–Witten monopole equations is presented. For mathematical progress on Seiberg–Witten monopoles, see Akbulut (1995) and Donaldson (1996) for the Abelian case and Bradlow and Garcia-Prada (1996) for the non-Abelian case.

First, let us recall the Seiberg–Witten monopole equations in four dimensions. We assume that X has Spin structure. Then there exist rank-two positive and negative spinor bundles $S^{\pm}$. For Abelian gauge theory, we introduce a complex line bundle L and a connection A_μ on L. The Weyl spinor M ($\overline{M}$) is a section of $S^+ \otimes L$ ($S^+ \otimes L^{-1}$), hence M satisfies the positive chirality condition $\gamma^5 M = M$. If X does not have Spin structure, we introduce Spin^c structure and Spin^c bundles $S^\pm \otimes L$, where L^2 is a line bundle. In this case, M should be interpreted as a section of $S^+ \otimes L$. Below, we assume Spin structure. The reader interested in the physical implications of Spin and Spin^c structures should refer to the excellent review by Thompson (1995) and references therein.

The Abelian Seiberg–Witten monopole equations (Witten, 1994a) in four dimensions are the set of following differential equations:

$$F^+_{\mu\nu} + \frac{i}{2}\overline{M}\sigma_{\mu\nu}M = 0$$

$$i\gamma^\mu D_\mu M = 0 \tag{3.1}$$

where $F^+_{\mu\nu}$ is the self-dual part of the $U(1)$ curvature tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F^+_{\mu\nu} = P^+_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{3.2}$$

and $P^+_{\mu\nu\rho\sigma}$ is the self-dual projector defined by

$$P^+_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\delta_{\mu\rho}\delta_{\nu\sigma} + \frac{\sqrt{g}}{2}\,\varepsilon_{\mu\nu\rho\sigma}\right) \tag{3.3}$$

Note that the second term in the first equation of (3.1) is also self-dual (Thompson, 1995). On the other hand, the second equation in (3.1) is a twisted Dirac equation whose covariant derivative D_μ is given by

$$D_\mu = \partial_\mu + \omega_\mu - iA_\mu \tag{3.4}$$

where

$$\omega_\mu = \frac{1}{4}\,\omega_\mu^{\alpha\beta}[\gamma_\alpha, \gamma_\beta] \tag{3.5}$$

is the spin connection 1-form on X.

In order to perform a reduction to three dimensions, let us first assume that X is a product manifold of the form $X = Y \times [0, 1]$, where Y is a three-dimensional compact manifold which has Spin structure. We may identify $t \in [0, 1]$ as a “time” variable, or we assume t as the zeroth coordinate of X, whereas x_i ($i = 1, 2, 3$) are the coordinates on (space manifold) Y. Then the metric is given by

$$ds^2 = dt^2 + g_{ij}\, dx^i\, dx^j \tag{3.6}$$

The dimensional reduction proceeds by assuming that all fields are independent of t. Below, we suppress the volume factor $\sqrt{g}$ of Y for simplicity.

First, let us consider the Dirac equation. After the dimensional reduction, the Dirac equation becomes

$$\gamma^i D_i M - i\gamma^0 A_0 M = 0 \tag{3.7}$$

As for the first monopole equation, using (3.2), we find that

$$\begin{aligned} F_{i0} + \frac{1}{2}\,\varepsilon_{i0jk} F_{i0} F^{jk} &= -i\overline{M}\sigma_{i0} M \\ F_{ij} + \varepsilon_{ijk0} F^{k0} &= -i\overline{M}\sigma_{ij} M \end{aligned} \tag{3.8}$$

Since the above two equations are dual to each other, the first one, for instance, can be reduced to the second one by a contraction with the totally antisymmetric tensor. Thus it is sufficient to consider one of them. Here, we take the first equation in (3.8).

After the dimensional reduction, (3.8) becomes

$$\partial_i A_0 - \frac{1}{2}\,\varepsilon_{ijk} F^{jk} = -i\overline{M}\sigma_{i0} M \tag{3.9}$$

where we have set $\varepsilon_{ijk} \equiv \varepsilon_{0ijk}$.

Therefore, the three-dimensional versions of the Seiberg–Witten equations are given by

$$\begin{aligned} \partial_i b - \frac{1}{2}\,\varepsilon_{ijk} F^{jk} + i\overline{M}\sigma_{i0} M &= 0 \\ i(\gamma^i D_i - i\gamma^0 b) M &= 0 \end{aligned} \tag{3.10}$$

where $b \equiv A_0$. The factor i of the Dirac equation is for later convenience.

It is now easy to establish the non-Abelian 3D monopole equations [for the four-dimensional version, see Labastida (1995), Labastida and Mariño, (1995b, c), Hyun *et al.* (1995a, b)] as

$$\begin{aligned} \partial_i b^a + f_{abc} A_i^b b^c - \frac{1}{2}\,\varepsilon_{ijk} F^{ajk} + i\overline{M}\sigma_{i0} T^a M &= 0 \\ i(\gamma^i D_i - i\gamma^0 b) M &= 0 \end{aligned} \tag{3.11}$$

where we have abbreviated $\overline{M}\sigma_{\mu\nu} T^a M \equiv \overline{M}^i \sigma_{\mu\nu} (T^a)_{ij} M^j$, subscripts to $(T^a)_{ij}$ run 1 to dim $\mathbf{g}$, and $b^a \equiv A_0^a$.

Next, let us find the action which produces (3.10). We can easily find that the simplest one is given by

$$S = \frac{1}{2}\int_Y \left[\left(\partial_i b - \frac{1}{2}\varepsilon_{ijk}F^{jk} + i\overline{M}\sigma_{i0}M\right)^2 + \left|i(\gamma^i D_i - i\gamma^0 b)M\right|^2\right] d^3x \tag{3.12}$$

Note that the minimum of (3.12) is given by (3.10). In this sense, the 3D monopole equations are not equations of motion, but constraints. Furthermore, there is a constraint for b. To see this, let us rewrite (3.12) as

$$S = \int_Y d^3x \left[\frac{1}{2}\left(\frac{1}{2}\varepsilon_{ijk}F^{jk} - i\overline{M}\sigma_{i0}M\right)^2 + \frac{1}{2}\left|\gamma^i D_i M\right|^2 + \frac{1}{4}(\partial_i b)^2 + \frac{1}{2}b^2\left|M\right|^2\right] \tag{3.13}$$

The minimum of this action is clear given by the 3D monopole equations with $b = 0$ for nontrivial A_i and M. However, for trivial A_i and M, we may relax the condition $b = 0$ to $\partial_i b = 0$, i.e., b is (in general) a nonzero constant. This can be also seen from (3.9). Accordingly, we obtain

$$\begin{aligned} \frac{1}{2}\varepsilon_{ijk}F^{jk} - i\overline{M}\sigma_{i0}M &= 0 \\ i\gamma^i D_i M &= 0 \\ b = 0 \quad \text{or} \quad \partial_i b &= 0 \end{aligned} \tag{3.14}$$

as an equivalent to (3.10), but we will use (3.10) for convenience. The Gaussian action will be used in the next section in order to construct a topological field theory by the Batalin–Vilkovisky quantization algorithm. The non-Abelian version of (3.12) and (3.14) would be obvious.

There is another action which can produce (3.14) as equations of motion. It is given by a Chern–Simons action coupled with matter (Kronheimer and Mrowka, 1994; Donaldson, 1996; Carey *et al.*, 1997), which is analogous to the action in massive gauge theory (Deser *et al.*, 1982), but we do not discuss the quantum field theory of this Chern–Simons action.

4. TOPOLOGICAL FIELD THEORIES OF 3D MONOPOLES

In this section, we construct topological field theories associated with the Abelian and non-Abelian 3D monopoles, using the Batalin–Vilkovisky quantization algorithm.

4.1. Abelian Case

A three-dimensional action for the Abelian 3D monopoles was found by the direct dimensional reduction of the four-dimensional one (Olsen, 1996; Carey *et al.*, 1997), but we show that the three-dimensional topological action can be also directly constructed from the 3D monopole equations.

4.1.1. Topological Action

A topological Bogomol'nyi action was constructed by using the Batalin–Vilkovisky quantization algorithm (Birmingham *et al.*, 1989) [a similar construction can be found in a two-dimensional version (Schaposnik and Thompson, 1989)] or by the quantization of magnetic charge (Baulieu and Grossman, 1988). The former is based on the quantization of a certain Langevin equation ("Bogomol'nyi monopole equation") and the classical action is quadratic, but the latter is based on the "quantization" of the pure topological invariant by using the Bogomol'nyi monopole equation as a gauge-fixing condition.

In order to compare the action to be constructed with those of Bogomol'nyi monopoles (Baulieu and Grossman, 1988; Birmingham *et al.*, 1989), we use the Batalin–Vilkovisky procedure. The reader unfamiliar with this construction may consult Batalin and Vilkovisky (1981, 1983, 1985), Labastida and Pernici (1988), Schaposnik and Thompson (1989), Birmingham *et al.* (1989, 1991), Gomis *et al.* (1995), and Hodges and Mohammedi (1996).

In order to obtain the topological action associated with 3D monopoles, we introduce random Gaussian fields G_i and ν ($\overline{\nu}$) and then start with the action

$$S_c = \frac{1}{2}\int_Y \left[\left(G_i - \partial_i b + \frac{1}{2}\varepsilon_{ijk}F^{jk} - i\overline{M}\sigma_{i0}M \right)^2 + \left|(\nu - i\gamma^i D_i M - \gamma^0 bM)\right|^2 \right] d^3x \tag{4.1}$$

Note that G_i and ν ($\overline{\nu}$) are also regarded as auxiliary fields. This action reduces to (3.12) in the gauge

$$G_i = 0, \qquad \nu = 0 \tag{4.2}$$

First, note that (4.1) is invariant under the topological gauge transformation

$$\delta A_i = \partial_i \theta + \varepsilon_i$$
$$\delta b = \tau$$
$$\delta M = i\theta M + \varphi$$

$$\delta G_i = \partial_i \tau - \varepsilon_{ijk}\partial^j \varepsilon^k + i(\overline{\varphi}\sigma_{i0} M + \overline{M}\sigma_{i0}\varphi)$$
$$\delta \nu = i\theta\nu + \gamma^i \varepsilon_i M + i\gamma^i D_i \varphi + \gamma^0 b\varphi + \gamma^0 \tau M \tag{4.3}$$

where θ is the parameter of gauge transformation, ε_i and $\tau \equiv \varepsilon_4$ are parameters which represent the topological shifts, and φ is the shift on the spinor space. Brackets for indices mean antisymmetrization, i.e.,

$$A_{[i}B_{j]} = A_i B_j - A_j B_i \tag{4.4}$$

Let us classify the gauge algebra (4.3). This is necessary in order to use the Batalin–Vilkovisky algorithm. Let us recall that the local symmetry for fields ϕ_i can be written generally in the form

$$\delta\phi_i = R^i_\alpha(\phi)\varepsilon^\alpha \tag{4.5}$$

where the indices represent the label of the fields and ε^α is a local parameter. When $\delta\phi_i = 0$ for nonzero ε^α, this symmetry is called first-stage-reducible. In the reducible theory, we can find zero-eigenvectors Z^α_a satisfying $R^i_\alpha Z^\alpha_a = 0$. Moreover, when the theory is on-shell-reducible, we can find such eigenvectors by using equations of motion.

For the case at hand, under the identifications

$$\theta = \Lambda, \qquad \varepsilon_i = -\partial_i \Lambda, \qquad \varphi = -i\Lambda M \tag{4.6}$$

and

$$\tau = 0 \tag{4.7}$$

(4.3) becomes

$$\delta A_i = 0$$
$$\delta b = 0$$
$$\delta M = 0$$
$$\delta G_i = 0$$
$$\delta\nu = i\Lambda(\nu - i\gamma^i D_i M - \gamma^0 b M)\big|_{\text{on-shell}} = 0 \tag{4.8}$$

Then for δA_i, for example, the R coefficients and the zero-eigenvectors are derived from

$$\delta A_i = R^{A_i}_\theta Z^\theta_\Lambda + R^{A_i}_{\varepsilon_j} Z^{\varepsilon_j}_\Lambda = 0 \tag{4.9}$$

that is,

$$R^{A_i}_\theta = \partial_i, \qquad R^{A_i}_{\varepsilon_j} = \delta_{ij}, \qquad Z^\theta_\Lambda = 1, \qquad Z^{\varepsilon_j}_\Lambda = -\partial_j \tag{4.10}$$

Of course, similar relations hold for other fields.

If we carry out BRST quantization via the Faddeev–Popov procedure in this situation, the Faddeev–Popov determinant will have zero modes. Therefore in order to fix the gauge further, we need a ghost for ghost. This reflects on the second-generation gauge invariance (4.8) realized on-shell. However, since b is irrelevant to Λ, the ghost for τ will not couple to the second-generation ghost. With this in mind, we use the Batalin–Vilkovisky algorithm in order to make the BRST quantization.

Let us assign new ghosts carrying opposite statistics to the local parameters. The assortment is given by

$$\theta \to c, \qquad \varepsilon_i \to \psi_i, \qquad \tau \to \xi, \qquad \varphi \to N \tag{4.11}$$

and

$$\Lambda \to \phi \tag{4.12}$$

Ghosts in (4.11) are first-generation, in particular, c is a Faddeev–Popov ghost, whereas ϕ is a second-generation ghost. Their Grassmann parity and ghost number (U number) are given by

$$\begin{matrix} c & \psi_i & \xi & N & \phi \\ 1^- & 1^- & 1^- & 1^- & 2^+ \end{matrix} \tag{4.13}$$

where the superscript on the ghost number denotes the Grassmann parity. Note that the ghost number counts the degree of the differential form on the moduli space $\mathcal{M}$ of the solution to the 3D monopole equations. The minimal set $\Phi_{\min}$ of fields consists of

$$\begin{matrix} A_i & b & M & G_i & \nu \\ 0^+ & 0^+ & 0^+ & 0^+ & 0^+ \end{matrix} \tag{4.14}$$

and (4.13).

On the other hand, the set of antifields $\Phi^*_{\min}$ carrying opposite statistics to $\Phi_{\min}$ is given by

$$\begin{matrix} A^*_i & b^* & M^* & G^*_i & \nu^* & c^* & \psi^*_i & N^* & \phi^* \\ -1^- & -1^- & -1^- & -1^- & -1^- & -2^+ & -2^+ & -2^+ & -3^- \end{matrix} \tag{4.15}$$

The next step is to find a solution to the master equation with $\Phi_{\min}$ and $\Phi^*_{\min}$, given by

$$\frac{\partial_r S}{\partial \Phi^A}\frac{\partial_l S}{\partial \Phi^*_A} - \frac{\partial_r S}{\partial \Phi^*_A}\frac{\partial_l S}{\partial \Phi^A} = 0 \tag{4.16}$$

where r (l) denotes the right (left) derivative.

The general solution for the first-stage-reducible theory at hand can be expressed by

$$S = S_c + \Phi_i^* R_\alpha^i C_1^\alpha + C_{1\alpha}^*(Z_\beta^\alpha C_2^\beta + T_{\beta\gamma}^\alpha C_1^\gamma C_1^\beta)$$
$$+ C_{2\gamma}^* A_{\beta\alpha}^\gamma C_1^\alpha C_2^\beta + \Phi_i^* \Phi_j^* B_\alpha^{ji} C_2^\alpha + \cdots \tag{4.17}$$

where C_1^α (C_2^α) denotes generally the first- (second-) generation ghost and only relevant terms in our case are shown. We often use $\Phi_{\min}^A = (\phi^i, C_1^\alpha, C_2^\beta)$, where ϕ^i denote generally the fields. In this expression, the indices should be interpreted as the label of the fields, and are not to be confused with space-time indices. The coefficients Z_β^α, $T_{\beta\gamma}^\alpha$, etc., can be directly determined from the master equation. In fact, it is known that these coefficients satisfy the following relations

$$R_\alpha^i Z_\beta^\alpha C_2^\beta - 2 \frac{\partial_r S_c}{\partial \phi^j} B_\alpha^{ji} C_2^\alpha (-1)^{|i|} = 0$$
$$\frac{\partial_r R_\alpha^i C_1^\alpha}{\partial \phi^j} R_\beta^j C_1^\beta + R_\alpha^i T_{\beta\gamma}^\alpha C_1^\gamma C_1^\beta = 0$$
$$\frac{\partial_r Z_\beta^\alpha C_2^\beta}{\partial \phi^j} R_\gamma^j C_1^\gamma + 2T_{\beta\gamma}^\alpha C_1^\gamma Z_\delta^\beta C_2^\delta + Z_\beta^\alpha A_{\delta\gamma}^\beta C_1^\gamma C_2^\delta = 0 \tag{4.18}$$

where $|i|$ means the Grassmann parity of the i th field.

In these expansion coefficients, R_α^i and Z_β^α are related to the local symmetry (4.3). On the other hand, as $T_{\beta\gamma}^\alpha$ is related to the structure constant of a given Lie algebra for a gauge theory, it is generally called a structure function. Of course, if the theory is Abelian, such a structure function does not appear. However, for a theory coupled with matter, all of the structure functions do not always vanish, even if the gauge group is Abelian. At first sight, this seems to be strange, but the expansion (4.17) obviously detects the coupling of matter fields and ghosts. In fact, the appearance of this type of structure function is required in order for the constructed action to be full BRST-invariant.

After some algebra, we will find the solution to be

$$S(\Phi_{\min}, \Phi_{\min}^*) = S_c + \int_Y \Delta S \, d^3x \tag{4.19}$$

where

$$\Delta S = A_i^*(\partial^i c + \psi^i) + b^* \xi + M^*(icM + N) + \overline{M}^*(-ic\overline{M} + \overline{N})$$
$$+ G_i^*[\partial^i \xi - \varepsilon^{ijk} \partial_j \psi_k + i(\overline{N} \sigma^{i0} M + \overline{M} \sigma^{i0} N)]$$

$$+ \nu^*(ic\nu + i\gamma^i D_i N + \gamma^i\psi_i M + \gamma^0 bN + \gamma^0\xi M)$$
$$+ \bar{\nu}^*\overline{(ic\nu + i\gamma^i D_i N + \gamma^i\psi_i M + \gamma^0 bN + \gamma^0\xi M)}$$
$$+ c^*\phi - \psi_i^*\partial^i\phi - iN^*(\phi M + cN) + \overline{iN}^*(\phi\bar{M} + c\bar{N})$$
$$+ 2i\nu^*\bar{\nu}^*\phi \tag{4.20}$$

We augment $\Phi_{\min}$ by new fields χ_i, d_i, μ ($\bar{\mu}$), ζ ($\bar{\zeta}$), λ, ρ, η, e, and the corresponding antifields. Their ghost number and Grassmann parity are given by

$$\begin{array}{cccccccc} \chi_i & d_i & \mu & \zeta & \lambda & \rho & \eta & e \\ -1^- & 0^+ & -1^- & 0^+ & -2^+ & -1^- & -1^- & 0^+ \end{array} \tag{4.21}$$

and

$$\begin{array}{cccc} \chi_i^* & \mu^* & \lambda^* & \rho^* \\ 0^+ & 0^+ & 1^- & 0^+ \end{array} \tag{4.22}$$

Then we look for the solution

$$S' = S(\Phi_{\min}, \Phi^*_{\min}) + \int_Y (\chi^{*i} d_i + \bar{\mu}^*\bar{\zeta} + \mu^*\zeta + \rho^* e + \lambda^*\eta)\, d^3x \tag{4.23}$$

where d_i, ζ, e, η are Lagrange multiplier fields.

In order to obtain the quantum action, we must fix the gauge. After a little thought, the best choice for the gauge-fixing condition which can reproduce the action obtained from the dimensional reduction of the four-dimensional one is found to be

$$\begin{aligned} G_i &= 0 \\ \nu &= 0 \\ \partial^i A_i &= 0 \\ -\partial^i\psi_i + \frac{i}{2}(\bar{N}M - \bar{M}N) &= 0 \end{aligned} \tag{4.24}$$

Thus we can obtain the gauge fermion carrying the ghost number -1 and odd Grassmann parity,

$$\Psi = -\chi^i G_i - \bar{\mu}\nu - \mu\bar{\nu} + \rho\partial^i A_i$$
$$- \lambda\left[-\partial^i\psi_i + \frac{i}{2}(\bar{N}M - \bar{M}N)\right] \tag{4.25}$$

The quantum action S_q can be obtained by eliminating antifields restricted to lie on the gauge surface

$$\Phi^* = \frac{\partial_r \Psi}{\partial \Phi} \tag{4.26}$$

Therefore the antifields will be

$$\begin{aligned}
&G_i^* = -\chi_i, \qquad \chi_i^* = -G_i, \qquad \nu^* = -\bar{\mu}, \\
&\bar{\nu}^* = -\mu, \qquad \bar{\mu}^* = -\nu, \qquad \mu^* = -\bar{\nu} \\
&M^* = -\frac{i}{2}\lambda\bar{N}, \qquad \bar{M}^* = \frac{i}{2}\lambda N, \qquad N^* = \frac{i}{2}\lambda\bar{M}, \qquad \bar{N}^* = -\frac{i}{2}\lambda M \\
&\rho^* = \partial^i A_i, \qquad A_i^* = -\partial_i\rho, \qquad \psi_i^* = -\partial_i\lambda \\
&\lambda^* = -\left[-\partial^i\psi_i + \frac{i}{2}(\bar{N}M - \bar{M}N)\right], \\
&c^* = \phi^* = b^* = \zeta^*\ (\bar{\zeta}^*) = 0
\end{aligned} \tag{4.27}$$

Then the quantum action S_q is given by

$$S_q = S'(\Phi, \Phi^* = \partial_r\Psi/\partial\Phi) \tag{4.28}$$

Substituting (4.27) into (4.28), we find that

$$S_q = S_c + \int_Y \tilde{\Delta} S\, d^3x \tag{4.29}$$

where

$$\begin{aligned}
\tilde{\Delta} S = {} & (-\Delta\phi + \phi\bar{M}M - i\bar{N}N)\lambda - \left[-\partial^i\psi_i + \frac{i}{2}(\bar{N}M - \bar{M}N)\right]\eta \\
& - \bar{\mu}(ic\nu + i\gamma^i D_i N + \gamma^i\psi_i M + \gamma^0 bN + \gamma^0\xi M) \\
& + \overline{(ic\nu + i\gamma^i D_i N + \gamma^i\psi_i M + \gamma^0 bN + \gamma^0\xi M)}\mu + 2i\phi\bar{\mu}\mu \\
& - \chi^i[\partial_i\xi - \varepsilon_{ijk}\partial^j\psi^k + i(\bar{N}\sigma_{i0}M + \bar{M}\sigma_{i0}N)] \\
& + \rho(\Delta c + \partial^i\psi_i) - d^i G_i - \bar{\zeta}\nu - \bar{\nu}\zeta + e\partial^i A_i
\end{aligned} \tag{4.30}$$

Using the condition (4.2) with $c = 0$, we arrive at

$$S_q' = S_c\Big|_{G_i=\nu(\bar{\nu})=0} + \int_Y \tilde{\Delta} S\Big|_{c=0}\, d^3x \tag{4.31}$$

where

$$\tilde{\Delta}S\Big|_{c=0} = (-\Delta\phi + \phi\overline{M}M - i\overline{N}N)\lambda - \left[-\partial^i\psi_i + \frac{i}{2}(\overline{N}M - \overline{M}N)\right]\eta$$
$$- \overline{\mu}(i\gamma^i D_i N + \gamma^i\psi_i M + \gamma^0 bN + \gamma^0\xi M)$$
$$+ \overline{(i\gamma^i D_i N + \gamma^i\psi_i M + \gamma^0 bN + \gamma^0\xi M)}\mu + 2i\phi\overline{\mu}\mu$$
$$- \chi^i[\partial_i\xi - \varepsilon_{ijk}\partial^j\psi^k + i(\overline{N}\sigma_{i0}M + \overline{M}\sigma_{i0}N)]$$
$$+ \rho\partial^i\psi_i + e\partial^i A_i \tag{4.32}$$

It is easy to find that (4.31) is consistent with the action found by the dimensional reduction of the four-dimensional topological action (Carey *et al.*, 1997).

4.1.2. BRST Transformation

The Batalin–Vilkovisky algorithm also facilitates the construction of the BRST transformation rule. The BRST transformation rule for a field Φ is defined by

$$\delta_B\Phi = \varepsilon \frac{\partial_r S'}{\partial\Phi^*}\Big|_{\Phi^*=\partial_r\Psi/\partial\Phi} \tag{4.33}$$

where ε is a constant Grassmann odd parameter. With this definition for (4.30), we obtain

$$\delta_B A_i = -\varepsilon(\partial_i c + \psi_i)$$
$$\delta_B b = -\varepsilon\xi$$
$$\delta_B M = -\varepsilon(icM + N)$$
$$\delta_B G_i = -\varepsilon[\partial_i\xi - \varepsilon_{ijk}\partial^j\psi^k + i(\overline{N}\sigma_{i0}M + \overline{M}\sigma_{i0}N)]$$
$$\delta_B\nu = -\varepsilon(ic\nu + i\gamma^i D_i N + \gamma^i\psi_i M + \gamma^0 bN + \gamma^0\xi M - i\mu\phi)$$
$$\delta_B c = \varepsilon\phi$$
$$\delta_B\psi_i = -\varepsilon\partial_i\phi$$
$$\delta_B\rho = \varepsilon e$$
$$\delta_B\lambda = -\varepsilon\eta$$
$$\delta_B\mu = \varepsilon\zeta$$
$$\delta_B N = -i\varepsilon(\phi M + cN)$$

$$\delta_B\chi_i = \varepsilon d_i$$
$$\delta_B\phi = \delta_B\xi = \delta_B d_i = \delta_B e = \delta_B\zeta = \delta_B\eta = 0 \tag{4.34}$$

It is clear at this stage that (4.34) has on-shell nilpotency, i.e., the quantum equation of motion for ν must be used in order to have $\delta_B^2 = 0$. This is due to the fact that the gauge algebra has on-shell reducibility. Accordingly, the Batalin–Vilkovisky algorithm gives a BRST-invariant action and on-shell nilpotent BRST transformation. Note that the equations

$$\partial_i\xi - \varepsilon_{ijk}\partial^j\psi^k + i(\overline{N}\sigma_{i0}M + \overline{M}\sigma_{i0}N) = 0$$
$$i\gamma^i D_i N + \gamma^i\psi_i M + \gamma^0 bN + \gamma^0\xi M = 0 \tag{4.35}$$

can be recognized as linearizations of the 3D monopole equations, and the number of linearly independent solutions gives the dimension of $\mathcal{M}$.

It is now easy to show that the global supersymmetry can be recovered from (4.34). In Witten-type theory, Q_B can be interpreted as a supersymmetric BRST charge. We define the supersymmetry transformation as

$$\delta_S\Phi := \delta_B\Phi\Big|_{c=0} \tag{4.36}$$

We can easily find that the result is consistent with the supersymmetry algebra of Carey *et al.* (1997).

4.1.3. Off-Shell Action

As was mentioned before, the quantum action of Witten-type topological field theory can be represented by a BRST commutator with nilpotent BRST charge Q_B. However, since our BRST transformation rule is on-shell nilpotent, we should integrate out ν and G_i in order to obtain the off-shell BRST transformation and off-shell quantum action.

For this purpose, let us consider the following terms in (4.30):

$$\frac{1}{2}(G_i - X_i)^2 + \frac{1}{2}\left|\nu - A\right|^2 - i\overline{\mu}c\nu + \overline{ic\nu}\mu - \overline{\zeta}\nu - \overline{\nu}\zeta \tag{4.37}$$
$$- d^iG_i$$

where

$$X_i = \partial_i b - \frac{1}{2}\varepsilon_{ijk}F^{jk} + i\overline{M}\sigma_{i0}M, \qquad A = i\gamma^i D_i M + \gamma^0 bM \tag{4.38}$$

Here, let us define

$$\nu' = \nu - A, \qquad B = -ic\mu - \zeta \tag{4.39}$$

ν' ($\bar{\nu}'$) and G_i can be integrated out and then (4.37) becomes

$$-\frac{1}{2}d_i d^i - d_i X^i - 2|B|^2 + \bar{B}A + B\bar{A} \tag{4.40}$$

Consequently, we obtain the off-shell quantum action

$$S_q = \{Q, \tilde{\Psi}\} \tag{4.41}$$

where

$$\begin{aligned}\tilde{\Psi} = &-\chi^i\left(X_i + \frac{\alpha}{2}d_i\right) - \mu\overline{(i\gamma^i D_i M + \gamma^0 bM - \beta B)}\\ &- \bar{\mu}(i\gamma^i D_i M + \gamma^0 bM - \beta B)\\ &+ \rho\partial^i A_i - \lambda\left[-\partial^i\psi_i + \frac{i}{2}(\bar{N}M - \bar{M}N)\right]\end{aligned} \tag{4.42}$$

α and β are arbitrary gauge-fixing parameters. A convenient choice for them is $\alpha = \beta = 1$. The BRST transformation rule for X_i and B fields can be easily obtained, although we do not write it down here.

4.1.4. Observables

We can now discuss the observables. For this purpose, let us define (Baulieu and Grossman, 1988)

$$\begin{aligned}\mathscr{A} &= A + c\\ \mathscr{F} &= F + \psi - \phi\\ \mathscr{H} &= db + \xi\end{aligned} \tag{4.43}$$

where we have introduced differential form notations, but their meanings should be obvious. A and c are considered as the (1, 0) and (0, 1) parts of the 1-form on $(Y, \mathscr{M})$. Similarly, F, ψ, and ϕ are the (2, 0), (1, 1), and (0, 2) parts of the 2-form $\mathscr{F}$, and db and ξ are the (1, 0) and (0, 1) parts of the 1-form $\mathscr{H}$. Thus $\mathscr{A}$ defines a connection 1-form on $(Y, \mathscr{M})$ and $\mathscr{F}$ is a curvature 2-form. Note that the exterior derivative d maps any (p_1, p_2)-form X_p of total degree $p = p_1 + p_2$ to a $(p_1 + 1, p_2)$-form, but δ_B maps any (p_1, p_2)-form to a $(p_1, p_2 + 1)$-form. Also note that

$$X_p X_q = (-1)^{pq} X_q X_p \tag{4.44}$$

Then the action of δ_B is

$$(d + \delta_B)\mathcal{A} = \mathcal{F}$$
$$(d + \delta_B)b = \mathcal{K} \tag{4.45}$$

$\mathcal{F}$ and $\mathcal{K}$ also satisfy

$$(d + \delta_B)\mathcal{F} = 0$$
$$(d + \delta_B)\mathcal{K} = 0 \tag{4.46}$$

Equations (4.46) can be interpreted as Bianchi identities in Abelian theory. Equations (4.45) and (4.46) represent the anticommuting property between the BRST variation δ_B and the exterior differential d, i.e., $\{\delta_B, d\} = 0$.

The BRST transformation rule in the geometric sector can be easily read off form (4.34), i.e., $\delta_B A$, $\delta_B\psi$, $\delta_B c$, and $\delta_B\phi$. Equations (4.46) imply

$$(d + \delta_B)\mathcal{F}^n = 0 \tag{4.47}$$

and expanding the above expression by ghost number and form degree, we obtain the following $(i, 2n - i)$-form $W_{n,i}$:

$$W_{n,0} = \frac{\phi^n}{n!}$$
$$W_{n,1} = \frac{\phi^{n-1}}{(n-1)!}\,\psi$$
$$W_{n,2} = \frac{\phi^{n-2}}{2(n-2)!}\,\Psi \wedge \psi - \frac{\phi^{n-1}}{(n-1)!}\,F$$
$$W_{n,3} = \frac{\phi^{n-3}}{6(n-3)!}\,\psi \wedge \psi \wedge \psi - \frac{\phi^{n-2}}{(n-2)!}\,F \wedge \psi \tag{4.48}$$

where

$$0 = \delta_B W_{n,0}$$
$$dW_{n,0} = \delta_B W_{n,1}$$
$$dW_{n,1} = \delta_B W_{n,2}$$
$$dW_{n,2} = \delta_B W_{n,3}$$
$$dW_{n,3} = 0 \tag{4.49}$$

Picking a certain k-cycle γ as a representative and defining the integral

$$W_{n,k}(\gamma) = \int_\gamma W_{n,k} \tag{4.50}$$

we can easily prove

$$\delta_B W_{n,k}(\gamma) = -\int_\gamma dW_{n,k-1} = -\int_{\partial\gamma} W_{n,k-1} = 0 \tag{4.51}$$

as a consequence of (4.49). Note that the last equality follows from the fact that the cycle γ is a simplex without boundary, i.e., $\partial\gamma = 0$. Therefore, $W_{n,k}(\gamma)$ indeed gives a topological invariant associated with the n th Chern class on $Y \times \mathcal{M}$.

On the other hand, since we have a scalar field b and its ghosts, we may construct topological observables associated with them. Therefore, the observables can be obtained from the ghost expansion of

$$(d + \delta_B)\mathcal{F}^n \wedge \mathcal{K}^m = 0 \tag{4.52}$$

Explicitly, for $m = 1$, for example, we obtain

$$\begin{aligned} 0 &= \delta_B W_{n,1,0} \\ dW_{n,1,0} &= \delta_B W_{n,1,1} \\ dW_{n,1,1} &= \delta_B W_{n,1,2} \\ dW_{n,1,2} &= \delta_B W_{n,1,3} \\ dW_{n,1,3} &= 0 \end{aligned} \tag{4.53}$$

where

$$\begin{aligned} W_{n,1,0} &= \frac{\phi^n}{n!}\xi \\ W_{n,1,1} &= \frac{\phi^{n-1}}{(n-1)!}\psi\xi - \frac{\phi^n}{n!}db \\ W_{n,1,2} &= \frac{\phi^{n-2}}{2(n-2)!}\psi \wedge \psi\xi - \frac{\phi^{n-1}}{(n-1)!}F\xi - \frac{\phi^{n-1}}{(n-1)!}\psi \wedge db \\ W_{n,1,3} &= \frac{\phi^{n-3}}{6(n-3)!}\psi \wedge \psi \wedge \psi\xi + \frac{\phi^{n-1}}{(n-1)!}F \wedge db \\ &\quad + \frac{\phi^{n-2}}{2(n-2)!}(2\psi \wedge F\xi + \psi \wedge \psi \wedge db) \end{aligned} \tag{4.54}$$

These correspond to the cocycles (Baulieu and Grossman, 1988) in the $U(1)$ case.

Next, let us look for the observables for the matter sector. The BRST transformation rules in this sector are given by $\delta_B M$, $\delta_B N$, $\delta_B c$, and $\delta_B \phi$. At first sight, the matter sector does not have any observable, but we can find that the combined form

$$\tilde{W} = i\phi \overline{M}M + \overline{N}N \tag{4.55}$$

is an observable. However, unfortunately, as $\tilde{W}$ is cohomologically trivial because $\delta_B \tilde{W} = 0$, then $d\tilde{W} \neq \delta_B \tilde{W}'$ for some $\tilde{W}'$. Accordingly, $\tilde{W}$ does not give any new topological invariant. Hyun *et al.* (1995a, b) identified $\tilde{W}$ as a part of the bare mass term of the hypermultiplet in their twisting construction of topological QCD in four dimensions.

In topological Bogomol'nyi theory, there is a sequence of observables associated with a magnetic charge. For the Abelian case, it is given by

$$W = \int_Y F \wedge db \tag{4.56}$$

As is pointed out for the case of Bogomol'nyi monopoles (Birmingham *et al.*, 1989), we cannot obtain the observables related to this magnetic charge by the action of δ_B as well, but we can construct those observables by the anti-BRST variation $\underline{\delta}_B$ which maps an (m, n)-form to an $(m, m-1)$-form. $\underline{\delta}_B$ can be obtained by a discrete symmetry which is realized as "time reversal symmetry" in four dimensions. In our three-dimensional theory, the discrete symmetry is given by

$$\phi \to -\lambda, \qquad \lambda \to -\phi, \qquad N \to i\sqrt{2}\overline{\mu}, \qquad \mu \to \frac{i}{\sqrt{2}}\overline{N}$$
$$\psi_i \to \frac{\chi_i}{\sqrt{2}}, \qquad \chi_i \to \sqrt{2}\overline{\psi}_i, \qquad \eta \to \sqrt{2}\overline{\xi}, \qquad \xi \to -\frac{\eta}{\sqrt{2}} \tag{4.57}$$

with

$$b \to -b \tag{4.58}$$

(4.58) is an additional symmetry (Birmingham *et al.*, 1989). Note that we must also change N and μ (and their conjugates). The positive chirality condition for M should be used to check the invariance of the action. In this way, we can obtain the anti-BRST transformation rule by substituting (4.57) and (4.58) into (4.34) and then we can obtain the observables associated with the magnetic charge by using the action of this anti-BRST variation (Birmingham *et al.*, 1989).

The topological observables available in this theory are the same as those of the topological Bogomol'nyi monopoles.

Finally, let us briefly comment on our three-dimensional theory. First note that the Lagrangian L and the Hamiltonian H in dimensional reduction can be considered as equivalent. This is because the relation between them is defined by

$$H = p\dot{q} - L \tag{4.59}$$

where q is any field, the overdot means time derivative, and p is the canonical conjugate momentum of q, and the dimensional reduction requires the time independence of all fields, thus $H = -L$ in this sense. Though we have constructed the three-dimensional action directly from the 3D monopole equations, our action may be interpreted essentially as the Hamiltonian of the four-dimensional Seiberg–Witten theory. In this sense (Witten, 1988), the ground states may correspond to the "Floer groups?" of Y, but we do not know the precise correspondence.

4.2. Non-Abelian Case

It is easy to extend the results obtained in the previous subsection to the non-Abelian case. In this subsection, we summarize the results for the non-Abelian 3D monopoles.

4.2.1. Non-Abelian Topological Action

With the auxiliary fields $G^a_{\mu\nu}$ and ν, we consider

$$S_c = \frac{1}{2}\int_Y d^3x \, [(G^a_i - K^a_i)^2 + \left|\nu - i\gamma^i D_i M - \gamma^0 bM\right|^2] \tag{4.60}$$

where

$$K^a_i = \partial_i b^a + f_{abc} A^b_i b^c - \frac{1}{2}\varepsilon_{ijk} F^a_{jk} + i\overline{M}\sigma_{i0} T^a M \tag{4.61}$$

Note that the minimum of (4.60) with the gauge

$$G^a_i = \nu = 0 \tag{4.62}$$

is given by the non-Abelian 3D monopoles. We take the generator of the Lie algebra in the fundamental representation, e.g., for $SU(n)$,

$$(T_a)_{ij}(T^a)_{kl} = \delta_{il}\delta_{jk} - \frac{1}{n}\delta_{ij}\delta_{kl} \tag{4.63}$$

Extension to other Lie algebras and representations is straightforward.

The gauge transformation rule for (4.60) is given by

$$\begin{aligned}
\delta A_i^a &= \partial_i\theta^a + f_{abc}A_i^b\theta^c + \varepsilon_i^a \\
\delta b^a &= f_{abc}b^b\theta^c + \tau^a \\
\delta M &= i\theta M + \varphi \\
\delta G_i^a &= f_{abc}G_i^b\theta^c + [-\varepsilon_{ijk}(\partial^j\varepsilon^{ak} + f_{abc}\varepsilon^{jb}A^{ck}) \\
&\quad + \partial_i\tau^a + f_{abc}(\varepsilon_i^b b^c - \tau^b A_i^c) + i(\overline{\varphi}\sigma_{i0}T^a M + \overline{M}\sigma_{i0}T^a\varphi)] \\
\delta\nu &= i\gamma^i D_i\varphi + \gamma^i\varepsilon_i M + \gamma^0 b\varphi + \gamma^0\tau M + i\theta\nu
\end{aligned} \tag{4.64}$$

Note that we have a G_i^a term in the transformation of G_i^a, while it did not appear in Abelian theory.

The gauge algebra (4.64) possesses on-shell zero modes as in the Abelian case. Setting

$$\theta^a = \Lambda^a, \qquad \varepsilon_i^a = -\partial_i\Lambda^a - f_{abc}A_i^b\Lambda^c, \qquad \tau^a = -f_{abc}b^b\Lambda^c, \qquad \varphi = -i\Lambda M \tag{4.65}$$

we can easily find that (4.64) closes

$$\begin{aligned}
\delta A_i^a &= 0 \\
\delta b^a &= 0 \\
\delta M &= 0 \\
\delta G_i^a &= f_{abc}\Lambda^c[G_i^b - K_i^b]\Big|_{\text{on-shell}} = 0 \\
\delta\nu &= i\Lambda[\nu - i(\gamma^i D_i - i\gamma^0 b)M]\Big|_{\text{on-shell}} = 0
\end{aligned} \tag{4.66}$$

when the equations of motion of G_i^a and ν are used. Note that we must use equations of motion of both G_i^a and ν in the non-Abelian case, while only "ν" was needed for the Abelian theory. Furthermore, as φ is a parameter in the spinor space, φ is not **g**-valued, in other words, $\varphi \neq \varphi^a T^a$. Equations (4.64) are first-stage-reducible.

The assortment of ghost fields, the minimal set $\Phi_{\min}$ of the fields and the ghost number and the Grassmann parity, and those for $\Phi^*_{\min}$ would be obvious.

Then the solution to the master equation is

$$S(\Phi_{\min}, \Phi^*_{\min}) = S_c + \int_Y \operatorname{tr} \Delta S_n \, d^3x \tag{4.67}$$

where

$$
\begin{aligned}
\Delta S_n = {} & A_i^*(D^i c + \psi^i) + b^*(i[b, c] + \xi) \\
& + M^*(icM + N) + \overline{M}^*(-ic\overline{M} + \overline{N}) \\
& + G^{*^i} \tilde{G}^i - iN^*(\phi M + cN) + i\overline{N}^*(\phi \overline{M} + c\overline{N}) \\
& + \nu^*(ic\nu + i\gamma^i D_i N + \gamma^i \psi_i M + \gamma^0 bN + \gamma^0 \xi M) \\
& + \overline{\nu}^* \overline{(ic\nu + i\gamma^i D_i N + \gamma^i \psi_i M + \gamma^0 bN + \gamma^0 \xi M)} \\
& + 2i\nu^* \overline{\nu}^* \phi + \psi_i^*(-D^i \phi - i\{\psi^i, c\}) \\
& + c^*\left(\phi - \frac{i}{2}\{c, c\}\right) - i\phi^*[\phi, c] \\
& - \frac{i}{2}\{G_i^*, G^{*^i}\}\phi + i\xi^*([b, \phi] - \{\xi, c\})
\end{aligned}
\tag{4.68}
$$

Here

$$
\begin{aligned}
\tilde{G}_i = {} & i[c, G_i] - \varepsilon_{ijk} D^j \psi^k + D_i \xi + [\psi_i, \xi] \\
& + i(\overline{N} \sigma_{i0} T_a T^a M + \overline{M} \sigma_{i0} T_a T^a N)
\end{aligned}
\tag{4.69}
$$

The equations

$$
\begin{aligned}
-\varepsilon_{ijk} D^j \psi^k + D_i \xi + [\psi_i, \xi] + i(\overline{N} \sigma_{i0} T_a T^a M + \overline{M} \sigma_{i0} T_a T^a N) &= 0 \\
i\gamma^i D_i N + \gamma^i \psi_i M + \gamma^0 bN + \gamma^0 \xi M &= 0
\end{aligned}
\tag{4.70}
$$

can be seen as linearizations of non-Abelian 3D monopoles.

We augment $\Phi_{\min}$ by new fields χ_i^a, d_i^a, μ ($\overline{\mu}$), ζ (ζ), λ, ρ, $\underline{\eta}$, e, and the corresponding antifields, but Lagrange multiplier fields d_i^a, ζ (ζ), e, and η are assumed not to have antifields for simplicity, and therefore their BRST transformation rules are set to zero. This simplification means that we do not take into account BRST exact terms. In this sense, the result to be obtained will correspond to those of the dimensionally reduced version of the four-dimensional theory (Labastida and Mariño, 1995b, c; Hyun *et al.*, 1995a, b) up to these terms, i.e., topological numbers.

From the gauge-fixing condition

$$
\begin{aligned}
G_i^a &= 0 \\
\nu &= 0 \\
\partial^i A_i &= 0 \\
-D^i \psi_i + \frac{i}{2}(\overline{N}M - \overline{M}N) &= 0
\end{aligned}
\tag{4.71}
$$

the gauge fermion becomes

$$\Psi = -\chi^i G_i - \overline{\mu}\nu - \mu\overline{\nu} + \rho\partial^i A_i - \lambda\left[-D^i\psi_i + \frac{i}{2}(\overline{N}M - \overline{M}N)\right] \tag{4.72}$$

The antifields are then given by

$$\begin{aligned}
&G_i^* = -\chi_i, \qquad \chi_i^* = -G_i, \qquad \nu^* = -\overline{\mu}, \\
&\overline{\nu}^* = -\mu, \qquad \overline{\mu}^* = -\nu, \qquad \mu = -\overline{\nu} \\
&M^* = -\frac{i}{2}\lambda\overline{N}, \qquad \overline{M}^* = \frac{i}{2}\lambda N, \qquad N^* = \frac{i}{2}\lambda\overline{M}, \qquad \overline{N}^* = -\frac{i}{2}\lambda M \\
&\rho^* = \partial^i A_i, \qquad A_i^* = -\partial_i\rho + i[\lambda, \psi_i], \qquad \psi_i^* = -D_i\lambda \\
&\lambda^* = -\left[-D_i\psi^i + [b, \xi] + \frac{i}{2}(\overline{N}M - \overline{M}N)\right] \\
&b^* = c^* = \xi^* = \phi^* = \zeta^*\ (\overline{\zeta}^*) = 0
\end{aligned} \tag{4.73}$$

Therefore we find the quantum action

$$S_q = S_c + \int_Y \operatorname{tr} \tilde{\Delta} S_n\, d^3x \tag{4.74}$$

where

$$\begin{aligned}
\tilde{\Delta} S_n = &-\left[-D_i\psi^i + [b, \xi] + \frac{i}{2}(\overline{N}M - \overline{M}N)\right]\eta - \lambda(D_i D^i\phi + iD_i\{\psi^i, c\}) \\
&+ i\lambda\{\psi_i, D^i c + \psi^i\} + (\phi\overline{M}M - i\overline{N}N)\lambda \\
&- \chi^i\Bigg[i[c, G_i] + \varepsilon_{ijk}D^j\psi^k + D_k\xi + [\psi_k, \xi] \\
&+ \frac{i}{2}(\overline{N}\sigma^{ij}T_aT^aM + \overline{M}\sigma^{ij}T_aT^aN)\Bigg] \\
&- \overline{\mu}(i\gamma^\mu D_\mu N + \gamma^\mu\psi_\mu M + ic\nu) + \overline{(i\gamma^i D_i N + \gamma^\mu\psi_\mu M + ic\nu)}\mu \\
&+ 2i\phi\overline{\mu}\mu - \frac{i}{2}\{\chi_i, \chi^i\}\phi + \rho(\partial_i D^i c + \partial_i\psi^i) \\
&- d^i G_i - \overline{\zeta}\nu - \overline{\nu}\zeta + e\partial^i A_i
\end{aligned} \tag{4.75}$$

In this quantum action, setting

$$M\,(\overline{M}) = N\,(\overline{N}) = \mu\;(\overline{\mu}) = \nu\;(\overline{\nu}) = 0 \tag{4.76}$$

we can find that the resulting action coincides with that of Bogomol'nyi monopoles (Birmingham *et al.*, 1989).

Finally, in order to obtain the off-shell quantum action, both auxiliary fields should be integrated out by a similar technique to that presented in the Abelian case, but we leave this as an exercise for the reader.

4.2.2. BRST Transformation

The BRST transformation rule is given by

$$\begin{aligned}
\delta_B A_i &= -\varepsilon(D_i c + \psi_i)\\
\delta_B b &= -\varepsilon(i[c, b] + \xi)\\
\delta_B \xi &= i\varepsilon([b, \phi] - \{\xi, c\})\\
\delta_B M &= -\varepsilon(icM + N)\\
\delta_B G_i &= -\varepsilon(\tilde{G}_i - i[\chi_i, \phi])\\
\delta_B \nu &= -\varepsilon(ic\nu + \gamma^\mu D_\mu N + \gamma^\mu \psi_\mu M - i\mu\phi)\\
\delta_B c &= \varepsilon\left(\phi - \frac{i}{2}\{c, c\}\right)\\
\delta_B \psi_i &= -\varepsilon(D_i\phi + i\{\psi_i, c\})\\
\delta_B \rho &= \varepsilon e\\
\delta_B \lambda &= -\varepsilon\eta\\
\delta_B \mu &= \varepsilon\zeta\\
\delta_B N &= -i\varepsilon(\phi M + cN)\\
\delta_B \chi_i &= \varepsilon d_i\\
\delta_B \phi &= i\varepsilon[\phi, c]\\
\delta_B d_i &= \delta_B e = \delta_B \zeta = \delta_B \eta = 0
\end{aligned} \tag{4.77}$$

It is easy to obtain supersymmetry also in this case. However, as we have omitted the BRST exact terms, the supersymmetry in our constructin does not detect them.

4.2.3. Observables

We have already constructed the topological observables for the Abelian case. In the non-Abelian case, the construction of observables is basically the same, except that relations (4.45) and (4.46) are modified,

$$(d + \delta_B)\mathcal{A} - \frac{i}{2}[\mathcal{A}, \mathcal{A}] = \mathcal{F}$$

$$(d + \delta_B)b - i[\mathcal{A}, b] = \mathcal{K} \tag{4.78}$$

and

$$(d + \delta_B)\mathcal{F} - i[\mathcal{A}, \mathcal{F}] = 0$$

$$(d + \delta_B)\mathcal{K} - i[\mathcal{A}, \mathcal{K}] = i[\mathcal{F}, b] \tag{4.79}$$

respectively, where $[*, *]$ is a graded commutator. The observables in the geometric and matter sectors are the same as before, but we should replace db by $d_A b$ in (4.54) as well as (4.78) and (4.79), where d_A is the exterior covariant derivative and trace is required. In addition, the magnetic charge observables are again obtained by anti-BRST variation as outlined before.

The observables in the geometric sector are those in (4.48) and follow the cohomology relation (4.49). In this way, the topological observables available in this three-dimensional theory are precisely the Bogomol'nyi monopole cocycles (Baulieu and Grossman, 1988).

5. SUMMARY

We have discussed the existence of topological field theories which describe the moduli space of Abelian and non-Abelian three-dimensional Seiberg–Witten monopole equations, by using the Batalin–Vilkovisky quantization procedure. In the Abelian case, our topological action with a certain gauge condition is found to be consistent with that of the dimensionally reduced version of the four-dimensional one. We have also established the three-dimensional non-Abelian action. The interesting point is that this non-Abelian action can be viewed as the Bogomol'nyi monopole topological action including matter and its associated ghost. We have easily obtained the BRST and anti-BRST transformation rules. The topological observables related to the Chern classes can be found in standard fashion. We have found that they are precisely the cocycles of Bogomol'nyi monopole topological field theory.

In this paper, we have not included the mass term for the Weyl spinor, but the introduction of the mass term may connect the Bogomol'nyi and the 3D Seiberg–Witten monopole topological field theory, as it was shown that the mass term interpolates Donaldson theory and Seiberg–Witten theory in four dimensions (Hyun *et al.*, 1995a, b). This point of view should be further studied.

While there has been progress on the self-dual Yang–Mills equations, several tasks remain for the Seiberg–Witten equations, so let us briefly comment on them as open problems.

1. *Integrability of Seiberg–Witten equations.* As is well known, the self-dual Yang–Mills equation can be reduced to certain solitonic equations such as the nonlinear Schrödinger equation or KdV equation after suitable choice for the gauge fields (Mason and Sparling, 1992; Strachan, 1993), although there is no proof that the self-dual Yang–Mills equation is indeed integrable. On the other hand, as for the Seiberg–Witten equations, they cannot be viewed as integrable equations at first sight, but it was found that the Seiberg–Witten equations on $\mathbf{R}^2$ could be realized as Liouville vortex equations, which are manifestly integrable (Nergiz and Saçlıoğlu, 1996) [as for a solution on $\mathbf{R}^3$, there is Freund's solution (Freund, 1995)]. This fact seems to connect integrable systems and Seiberg–Witten monopoles. Furthermore, as explicit solutions to non-Abelian Seiberg–Witten equations have not been found, we cannot pursue the integrability. For this direction, the twistor program (Penrose and MacCallu, 1972) may be available, as is often used for the self-dual Yang–Mills equation (Corrigan and Goddard, 1981; Ward, 1981; Mason and Sparling, 1992).

2. *Reduction to two-dimensional surfaces* (Riemann surfaces Σ). We can dimensionally reduce the Seiberg–Witten equations onto two-dimensional surfaces. As mentioned before, the operation of dimensional reduction connects the theories between four and three, and three and two dimensions; the two-dimensional theory may be regarded as a dual $U(1)$ theory for the $SU(2)$ Hitchin equations (Hitchin, 1987) [usually, the Lie group for Hitchin equations is taken to be $SO(3)$ rather than $SU(2)$], i.e., two-dimensional Yang–Mills–Higgs equations. One approach to study this observation is to construct solutions to the Seiberg–Witten equations on Riemann surfaces and compare their properties with those of the Hitchin equations. Recently, the reduced Seiberg–Witten equations were studied and it was pointed out that the set of equations had an extremely similar structure to the Hitchin equations, except for the distinction of Higgs field and Weyl spinor (Martin and Restuccia, 1997). We would like to interpret the relationship between these two theories in the context of topological quantum field theory, but no progress such as the study of the topological field theory associated with the two-dimensional Seiberg–Witten monopoles has been made [a topological action is obtained by a dimensional reduction (Olsen, 1996)], although Yang–Mills theory on Riemann surfaces are well discussed (see e.g., Thompson, 1995,

and references therein). We know that the Yang–Mills–Higgs theory in two dimensions is closely related to a conformal field theory (Chapline and Grossman, 1989), but is it true also in two-dimensional Seiberg–Witten theory?

There are other problems, such as supersymmetric extension (Ader *et al.*, 1989; Ader and Grieres, 1990), twistor description (Volovich, 1983; Nair and Schiff, 1989), and so on, but many (topological) field-theoretic techniques to study these problems have been developed. Nevertheless, the Seiberg–Witten theory does not seem to have been discussed even in four dimensions as well as lower dimensions, in contrast with the Donaldson theory in the context of topological quantum field theory. Filling the gap may be an attractive problem, but much effort will be required.

APPENDIX. CONVENTION FOR GAMMA MATRIX

The convention for the gamma matrix is as follows. Let σ^i be Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.1}$$

On $\mathbf{R}^4$ we define four gamma matrices

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & i\sigma^j \\ -i\sigma^j & 0 \end{pmatrix} \tag{A.2}$$

and

$$\begin{aligned} \gamma^5 &= \gamma^0\gamma^1\gamma^2\gamma^3 \\ &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \end{aligned} \tag{A.3}$$

where I is a 2×2 unit matrix.

As can be easily seen from (A.2), they satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta_{\mu\nu} \tag{A.4}$$

It is often useful to define

$$\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] \tag{A.5}$$

Then

$$\sigma_{ij} = i\varepsilon_{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \qquad \sigma_{k0} = i\begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix} \tag{A.6}$$

On curved manifolds, we multiply viervein to these gamma matrices (except γ^5). On $Y \times [0, 1]$, γ^0 is a constant matrix, while the others are not in general.

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